Radboud University Nijmegen

# Dold-Kan Correspondence

Bachelor Thesis in Mathematics

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## Introduction

In this thesis we will study the Dold-Kan correspondence, a celebrated result which belongs to the field of homological algebra or simplicial homotopy theory. Abstractly, one version of the theorem states that there is an equivalence of categories

$$K : \mathbf{Ch}(\mathbf{Ab}) \simeq \mathbf{sAb} : N,$$

where $\mathbf{Ch}(\mathbf{Ab})$ is the category of chain complexes and $\mathbf{sAb}$ is the category of simplicial abelian groups. This theorem was discovered by A. Dold [**Dol58**] and D. Kan [**Kan58**] independently in 1957. Objects of either of these categories have important invariants. A more refined statement of this equivalence tells us that there is a natural isomorphism between homology groups of chain complexes and homotopy groups of simplicial abelian groups. A bit more precise:

$$\pi_n(A) \cong H_n(N(A)) \text{ for all } n \in \mathbb{N}.$$

In the first section some definitions from category theory are recalled, which are especially important in Sections 3 and 4. In Section 2 we will discuss the category of chain complexes and in the end of this section a motivation from algebraic topology will be given for these objects. Section 3 then continues with the other category involved, the category of simplicial abelian groups. This section starts with a slightly more general notion and it will be illustrated to have a geometrical meaning. In Section 4 the correspondence will be defined and proven. In the last section (Section 5) the refined statement will be proven and in the end some more general notes about topology and homotopy will be given, justifying once more the beauty of this correspondence.

## 1. Category Theory

Before we will introduce the two categories $\mathbf{Ch}(\mathbf{Ab})$ and $\mathbf{sAb}$, let us begin by recalling some basic category theory. The reader who is already familiar with these concepts, is invited to skip this section. We will recall the notions of categories, functors, isomorphisms, natural transformations, equivalences, adjunctions and the Yoneda lemma.

We will briefly define categories and functors to fix the notation. We will not provide many examples or intuition in these concepts. For a more elaborated exposition one should have a read in [**Awo10**] or [**ML98**]. The more complicated definitions will be discussed in a bit more detail.

### 1.1. Categories.

Definition 1.1. A *category* $\mathbf{C}$ consists of a collection of *objects*, a set of *maps* $\mathbf{Hom}_{\mathbf{C}}(A, B)$ for each two objects $A, B \in \mathbf{C}$ and a binary operator *composition*

$$- \circ - : \mathbf{Hom}_{\mathbf{C}}(B, C) \times \mathbf{Hom}_{\mathbf{C}}(A, B)$$

such that

- composition is associative, i.e. $h \circ (g \circ f) = (h \circ g) \circ f$ and
- there exists an neutral element $\mathbf{id}_A \in \mathbf{Hom}_{\mathbf{C}}(A, A)$ for all $A$ in $\mathbf{C}$, i.e.
$$\mathbf{id}_B \circ f = f = f \circ \mathbf{id}_A.$$

Instead of writing $f \in \mathbf{Hom}_{\mathbf{C}}(A, B)$ we write $f : A \to B$, as many categories have functions as maps. For brevity we sometimes write $gf$ instead of $g \circ f$. We will need the category $\mathbf{Set}$ of sets with functions, the category $\mathbf{Ab}$ of abelian groups with group homomorphisms and the category $\mathbf{Top}$ of topological spaces and continuous maps.

Definition 1.2. A *functor* $F$ from a category $\mathbf{C}$ to a category $\mathbf{D}$ consists of a function $F_0$ from the objects of $\mathbf{C}$ to the objects of $\mathbf{D}$ and a function $F_1$ from maps in $\mathbf{C}$ to maps in $\mathbf{D}$, such that

- for $f : A \to B$, we have $F_1(f) : F_0(A) \to F_0(B)$,
- $F_1(\mathbf{id}_A) = \mathbf{id}_{F_0(A)}$ and
- $F_1(g \circ f) = F_1(g) \circ F_1(f)$.

We normally do not write the index of $F_0$ or $F_1$, instead we write $F$ for both functions.

For a category $\mathbf{C}$ we denote the *opposite* category by $\mathbf{C}^{op}$. The opposite category consists of the same objects, but the maps and composition are reversed. A *contravariant functor* $F$ from $\mathbf{C}$ to $\mathbf{D}$ is a functor $F : \mathbf{C}^{op} \to \mathbf{D}$.

Note that the composition of two functors is again a functor, and that we always have an identity functor, sending each object to itself and each map to itself. This gives rise to a category $\mathbf{Cat}$ of *small* categories. Note that we need some kind of *smallness* to avoid set-theoretical issues. However we will not be interested in these set-theoretical issues, and hence skip the definition of small.

**1.2. Isomorphisms.** Given a category $\mathbf{C}$ and two objects $A, B \in \mathbf{C}$ we would like to know when those objects are regarded as the same, according to the category. This will be the case when there is an isomorphism between the two.

Definition 1.3. A map $f : A \to B$ in a category $\mathbf{C}$ is an *isomorphism* if there is a map $g : B \to A$ such that
$$f \circ g = \mathbf{id}_B \quad \text{and} \quad g \circ f = \mathbf{id}_A.$$

Isomorphisms in $\mathbf{Ab}$ are exactly the isomorphisms which we know, i.e. the group homomorphisms which are both injective and surjective. For example the cyclic group $\mathbb{Z}_4$ and the Klein four-group $V_4$ are not isomorphic in $\mathbf{Ab}$, but if we regard only the sets $\mathbb{Z}_4$ and $V_4$, then they are (because there is a bijection). So it is good to note that whether two objects are isomorphic really depends on the category we are working in.

Note that an isomorphism between two categories is now also defined. Two categories $\mathbf{C}$ and $\mathbf{D}$ are isomorphic if there are functors $F$ and $G$ such that $FG = \mathbf{id}_\mathbf{D}$ and $GF = \mathbf{id}_\mathbf{C}$.

**1.3. Natural transformations.**

Definition 1.4. Given two functors $F, G : \mathbf{C} \to \mathbf{D}$, a *natural transformation* $\phi$ from $F$ to $G$, is a family of maps $\phi_c : F(c) \to G(c)$ for $c \in \mathbf{C}$, such that

$$\begin{tikzcd} F(c) \arrow[r, "\phi_c"] \arrow[d, "F(f)"] & G(c) \arrow[d, "G(f)"] \\ F(c') \arrow[r, "\phi_{c'}"] & G(c') \end{tikzcd}$$

commutes for any map $f : c \to c'$ and any objects $c, c' \in \mathbf{C}$.

For any two categories $\mathbf{C}$ and $\mathbf{D}$ we can form a category with functors $F : \mathbf{C} \to \mathbf{D}$ as objects and natural transformations as maps. This category is called the *functor category* and is denoted by $\mathbf{D}^\mathbf{C}$.

This now also gives a notion of isomorphisms between functors. It can be easily seen that an isomorphism between two functors is a natural transformation which is an isomorphism pointwise. Such a natural transformation is called a *natural isomorphism*.

For any category $\mathbf{C}$ we can define the $\mathbf{Hom}$-functor. It assigns to two objects in $\mathbf{C}$ the set of maps between them, i.e.
$$\mathbf{Hom}_\mathbf{C}(-,-) : \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{Set}.$$
We will show that it indeed defines a functor in the first argument, a similar proof works for the second argument. Let $f : A' \to A$ be a map in $\mathbf{C}$ and $g \in \mathbf{Hom}_\mathbf{C}(A, B)$, then $g \circ f \in \mathbf{Hom}_\mathbf{C}(A', B)$. Hence the assignment $g \mapsto g \circ f$ is a map from $\mathbf{Hom}_\mathbf{C}(A, B)$ to $\mathbf{Hom}_\mathbf{C}(A', B)$. Note that the direction of the map if reversed. Using associativity and identity it is easily checked that this assignment is functorial.

**1.4. Equivalence.** Recall that an *isomorphism* between categories $\mathbf{C}$ and $\mathbf{D}$ consists of two functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ such that

$$FG = \mathbf{id_D} \quad \text{and} \quad \mathbf{id_C} = GF.$$

With the notion of isomorphisms between functors we can generalize this, and only require a natural isomorphism instead of equality.

Definition 1.5. An *equivalence* between two categories $\mathbf{C}$ and $\mathbf{D}$ consists of two functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ such that there are natural isomorphisms:

$$FG \cong \mathbf{id_D} \quad \text{and} \quad \mathbf{id_C} \cong GF.$$

This is denoted by $\mathbf{C} \simeq \mathbf{D}$.

Example 1.6. The category $\mathbf{Set_{fin}}$ of finite sets is equivalent to the category $\mathbf{Ord_{fin}}$ of finite ordinals (with all functions). The former is uncountable and the latter is countable, hence they clearly cannot be isomorphic. However, from a categorical point of view these categories are very alike, which is precisely expressed by the equivalence.

**1.5. Adjunctions.**

Definition 1.7. An *adjunction* between two categories $\mathbf{C}$ and $\mathbf{D}$ consists of two functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ together with a natural bijection

$$\mathbf{Hom_D}(FX, Y) \cong \mathbf{Hom_C}(X, GY),$$

for any $X \in \mathbf{C}$ and $Y \in \mathbf{D}$. The functor $F$ is called the *left adjoint* and $G$ the *right adjoint*.

There are different equivalent descriptions of adjunctions. A particular nice one will be recalled. For a proof of equivalence to the above definition we refer to books on category theory such as the one of Mac Lane [**ML98**] or Awodey [**Awo10**].

Lemma 1.8. *Given functors $F : \mathbf{C} \to \mathbf{D}$, $G : \mathbf{D} \to \mathbf{C}$ then $F$ is a left adjoint and $G$ a right adjoint if and only if there exists a natural transformation, called the* unit

$$\eta : \mathbf{id_C} \to GF.$$

*such that for any map $f : S \to G(A)$ (in $\mathbf{C}$), there is a unique map $\overline{f} : F(S) \to A$ (in $\mathbf{D}$) such that $G(\overline{f}) \circ \eta = f$. I.e.:*

$$\begin{tikzcd} S \arrow[r, "\eta"] \arrow[dr, "f"'] & GF(S) \arrow[d, "G(\overline{f})"] \\ & G(A) \end{tikzcd} \qquad \begin{tikzcd} F(S) \arrow[d, "\overline{f}"] \\ A \end{tikzcd}$$

Note that by considering the identity map $\mathbf{id} : G(A) \to G(A)$ in $\mathbf{C}$, we get a uniquely determined map $\overline{\mathbf{id}} : FG(A) \to A$. This map $FG(A) \to A$ is in fact natural in $A$, this natural transformation is called the *co-unit*

$$\varepsilon : FG \to \mathbf{id}.$$

It can be shown that an equivalence $F : \mathbf{C} \xrightarrow{\simeq} \mathbf{D}$ is both a left and right adjoint. We sketch the proof of $F$ being a left adjoint. Clearly we already have the natural transformation $\eta : \mathbf{id_C} \to GF$. To construct $\overline{f}$ from $f : S \to G(A)$ we can apply the functor $F$, to get $F(S) \to FG(A)$, using the other natural isomorphism we get $F(S) \to FG(A) \to A$. We leave the details to the reader.

The first definition of adjunction is useful when dealing with maps, since it gives an bijection between the **Hom**-sets. However the second definition is useful when proving a certain construction is part of an adjunction, as shown in the following example.

Example 1.9. *(Free abelian groups)* There is an obvious functor $U : \mathbf{Ab} \to \mathbf{Set}$, which sends an abelian group to its underlying set, forgetting the additional structure. It is hence called a *forgetful functor*. This functor has a left adjoint $\mathbb{Z}[-] : \mathbf{Set} \to \mathbf{Ab}$ given by the *free abelian group functor*. For a set $S$ define

$$\mathbb{Z}[S] = \{\phi : S \to \mathbb{Z} \mid \mathrm{supp}(\phi) \text{ is finite}\},$$

where $\mathrm{supp}(\phi) = \{s \in S \mid \phi(s) \neq 0\}$. The group structure on $\mathbb{Z}[S]$ is given by pointwise addition. We can define a generator $e_s \in \mathbb{Z}[S]$ for every element $s \in S$ as

$$e_s(t) = \begin{cases} 1 \text{ if } s = t \\ 0 \text{ otherwise} \end{cases}.$$

One can think of elements of this abelian group as formal sums, namely by writing $\phi \in \mathbb{Z}[S]$ as $\phi = \sum_{x \in \mathrm{supp}(\phi)} \phi(x) e_x$. In other words $\mathbb{Z}[S]$ consists of linear combinations of elements in $S$. The functor $\mathbb{Z}[-]$ is defined on functions as follows. Let $f : S \to T$ be a function, then define

$$\mathbb{Z}[f](\phi) = \sum_{x \in \mathrm{supp}(\phi)} \phi(x) e_{f(x)} \quad \text{for all } \phi \in \mathbb{Z}[S].$$

It is left for the reader to check that this indeed gives a group homomorphism and that the functor laws hold. There is a map $\eta : S \to U\mathbb{Z}[S]$ given by

$$\eta(s) = e_s.$$

And given any map $f : S \to U(A)$ for any abelian group $A$, we can define

$$\overline{f}(\phi) = \sum_{x \in \mathrm{supp}(\phi)} \phi(x) \cdot e_{f(x)}.$$

It is clear that $U(\overline{f}) \circ \eta = f$. We will leave the other details (naturality of $\eta$, $\overline{f}$ being a group homomorphism, and uniqueness w.r.t. $U(\overline{f}) \circ \eta = f$) to the reader.

By the other description of adjunctions we have $\mathbf{Hom_{Ab}}(\mathbb{Z}[S], A) \cong \mathbf{Hom_{Set}}(S, U(A))$, which exactly tells us that we can define a group homomorphism from $\mathbb{Z}[S]$ to $A$ by only specifying it on the generators $e_s, s \in S$. This fact is used throughout this thesis.

**1.6. The Yoneda lemma.** So far we have only encountered definitions from category theory. However there is a very important lemma by Yoneda. This lemma gives a nice way to construct certain natural transformations.

DEFINITION 1.10. For any category $\mathbf{C}$, we define a functor $y : \mathbf{C} \to \mathbf{Set}^{\mathbf{C}^{op}}$ as follows

$$y(X) = \mathbf{Hom}_{\mathbf{C}}(-, X).$$

The functor $y$ is called the *Yoneda embedding.*

We will denote the set of natural transformation between two functors $F, G : \mathbf{C} \to \mathbf{D}$ as

$$\mathbf{Nat}(F, G) = \mathbf{Hom}_{\mathbf{D}^{\mathbf{C}}}(F, G).$$

LEMMA 1.11. (The Yoneda lemma) *Given a functor $F : \mathbf{C} \to \mathbf{Set}$ and any object $C \in \mathbf{C}$ there is a bijection*

$$\mathbf{Nat}(y(C), F) \cong F(C),$$

*which is natural in both $F$ and $C$.*

We will not provide a proof of this lemma, but we will give the function which can be proven to be a natural bijection. Given a natural transformation $\phi \in \mathbf{Nat}(y(C), F)$, we can consider the map $\phi_C : y(C)(C) \to F(C)$. Note that the codomain already is the right set, we only have to apply $\phi_C$ to the right object. The bijection is given by

$$\phi \mapsto \phi_C(\mathbf{id}_C).$$

We will use this lemma when we discuss simplicial abelian groups.

## 2. Chain Complexes

DEFINITION 2.1. A *(non-negative) chain complex of abelian groups* $C$ is a collection of abelian groups $C_n$, $n \in \mathbb{N}$, together with group homomorphisms $\partial_n : C_n \to C_{n-1}$, which we call *boundary operators*, such that $\partial_n \circ \partial_{n+1} = 0$ for all $n \in \mathbb{N}^{>0}$.

Thus graphically a chain complex $C$ can be depicted by the following diagram:

$$\cdots \xrightarrow{\partial_5} C_4 \xrightarrow{\partial_4} C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

There are many variants to this notion. For example, there are also unbounded chain complexes with an abelian group for each $n \in \mathbb{Z}$ instead of $\mathbb{N}$. In this thesis we will only need chain complexes in the sense of the definition above. Hence we will simply call them chain complexes, instead of non-negative chain complexes. Other variants can be given by taking a collection of $R$-modules instead of abelian groups. Of course not any kind of mathematical object will suffice, because we need to be able to express $\partial_n \circ \partial_{n+1} = 0$, so we need some kind of *zero object*. We will not need this kind of generality and stick to abelian groups.

In order to organize these chain complexes in a category, we should define what the maps are. The diagram above already gives an idea for this.

DEFINITION 2.2. Let $C$ and $D$ be chain complexes, with boundary operators $\partial_n^C$ and $\partial_n^D$ respectively. A *chain map* $f : C \to D$ consists of a family of group homomorphisms $f_n : C_n \to D_n$, such that they commute with the boundary operators: $f_n \circ \partial_{n+1}^C = \partial_{n+1}^D \circ f_{n+1}$ for all $n \in \mathbb{N}$, i.e. the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_5^C"] & C_4 \arrow[r, "\partial_4^C"] \arrow[d, "f_4"] & C_3 \arrow[r, "\partial_3^C"] \arrow[d, "f_3"] & C_2 \arrow[r, "\partial_2^C"] \arrow[d, "f_2"] & C_1 \arrow[r, "\partial_1^C"] \arrow[d, "f_1"] & C_0 \arrow[d, "f_0"] \\
\cdots \arrow[r, "\partial_5^D"] & D_4 \arrow[r, "\partial_4^D"] & D_3 \arrow[r, "\partial_3^D"] & D_2 \arrow[r, "\partial_2^D"] & D_1 \arrow[r, "\partial_1^D"] & D_0
\end{tikzcd}$$

Note that if we have two such chain maps $f : C \to D$ and $g : D \to E$, then the level-wise composition will give us a chain map $g \circ f : C \to D$. Also taking the identity function in each degree, gives us a chain map $\mathbf{id} : C \to C$. In fact, this will form a category, we will leave the details (the identity law and associativity) to the reader.

DEFINITION 2.3. $\mathbf{Ch}(\mathbf{Ab})$ is the category of chain complexes of abelian groups with chain maps.

Note that we will often drop the indices of the boundary operators, since it is often clear in which degree we are working. The boundary operators give rise to certain subgroups, because all groups are abelian, subgroups are normal subgroups.

DEFINITION 2.4. Given a chain complex $C$ we define the following subgroups:

- the subgroup of *$n$-cycles*: $Z_n(C) = \ker(\partial_n : C_n \to C_{n-1}) \trianglelefteq C_n$,
- the subgroup of *0-cycles*: $Z_0(C) = C_0$, and
- the subgroup of *$n$-boundaries*: $B_n(C) = \mathrm{im}(\partial_{n+1} : C_{n+1} \to C_n) \trianglelefteq C_n$.

LEMMA 2.5. *Given a chain complex $C$ we have for all $n \in \mathbb{N}$:*

$$B_n(C) \trianglelefteq Z_n(C).$$

PROOF. It follows from $\partial_n \circ \partial_{n+1} = 0$ that $\mathrm{im}(\partial : C_{n+1} \to C_n)$ is a subset of $\ker(\partial : C_n \to C_{n-1})$. Those are exactly the abelian groups $B_n(C)$ and $Z_n(C)$, so $B_n(C) \trianglelefteq Z_n(C)$. □

In general there is no inclusion in the other direction. This defect can be measured by a quotient and gives rise to the following definition. A motivation for this concept will be provided in Section 2.2.

DEFINITION 2.6. Given a chain complex $C$ we define the *$n$-th homology group $H_n(C)$* for each $n \in \mathbb{N}$ as:

$$H_n(C) = Z_n(C)/B_n(C).$$

We will denote the class of an $n$-cycle $x \in Z_n(C)$ by $[x]$ and refer to it as the *homology class of $x$*.

LEMMA 2.7. *The $n$-th homology group gives a functor $H_n : \mathbf{Ch}(\mathbf{Ab}) \to \mathbf{Ab}$ for each $n \in \mathbb{N}$.*

PROOF. Let $f : C \to D$ be a chain map and $n \in \mathbb{N}$. First note that for $x \in Z_n(C)$ we have $\partial^C(x) = 0$, so $\partial^D(f_n(x)) = 0$, because the square on the right commutes:

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\partial^C} & C_{n+1} & \xrightarrow{\partial^C} & C_n & \xrightarrow{\partial^C} & C_{n-1} & \xrightarrow{\partial^C} & \cdots \\
 & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\cdots & \xrightarrow{\partial^D} & D_{n+1} & \xrightarrow{\partial^D} & D_n & \xrightarrow{\partial^D} & D_{n-1} & \xrightarrow{\partial^D} & \cdots
\end{array}$$

So there is an induced group homomorphism $f_n^Z : Z_n(C) \to Z_n(D)$ (for $n = 0$ this is trivial). Similarly there is an induced group homomorphism $f_n^B : B_n(C) \to B_n(D)$ by considering the square on the left. Now define the map $H_n(f) : [x] \mapsto [f_n(x)]$ for $x \in Z_n(C)$, we now know that $f_n(x)$ is also a cycle, because of $f_n^Z$. Furthermore it is well-defined on classes, because of $f_n^B$. So indeed there is an induced group homomorphism $H_n(f) : H_n(C) \to H_n(D)$.

It remains to check that $H_n$ preserves identities and compositions. By writing out the definition we see $H_n(\mathbf{id})([x]) = [\mathbf{id}(x)] = [x] = \mathbf{id}[x]$, and:

$$H_n(g \circ f)([x]) = [g_n(f_n(x))] = H_n(g)([f_n(x)]) = H_n(g) \circ H_n(f)([x]).$$

□

**2.1. A note on abelian categories.** The category $\mathbf{Ch}(\mathbf{Ab})$ in fact is an *abelian category.* We will only need a very specific property of this fact later on, and hence we will only prove this single fact. For the precise definition of an abelian category we refer to the book of Rotman about homological algebra [**Rot09**, Chapter 5.5]. The notion of an abelian category is interesting if one wants to consider chain complexes over other objects than abelian groups, because $\mathbf{Ch}(\mathbf{C})$ will be an abelian category whenever $\mathbf{C}$ is an abelian category.[1] The property we want to use later on is the following.

Definition 2.8. A category $\mathbf{C}$ is *preadditive* if the set of maps between two objects is an abelian group, such that composition is bilinear. In other words: the **Hom**-functor has as its codomain $\mathbf{Ab}$:

$$\mathbf{Hom}_{\mathbf{C}}(-,-) : \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{Ab}.$$

To see why functoriality is the same as bilinear composition, recall that the **Hom**-functor in the first variable uses precomposition on maps, and postcomposition in the second variable. By functoriality this should be a group homomorphism, written out this means: $h \circ (g + f) = h \circ g + h \circ f$ for postcomposition, in other words postcomposition is linear. Similar for precomposition. Together this gives bilinearity of $- \circ -$.

Clearly the category $\mathbf{Ab}$ is preadditive, since we can add group homomorphisms pointwise. Furthermore, postcomposition is linear $h \circ (g+f)(x) = h(g(x)+f(x)) = h(g(x)) + h(f(x)) = (h \circ g + h \circ f)(x)$, and similarly precomposition is linear. Using this we can proof the following.

Lemma 2.9. *The category* $\mathbf{Ch}(\mathbf{Ab})$ *is a preadditive category.*

Proof. We can add chain maps level-wise. Given two chain maps $f, g : C \to D$, we define $f + g$ as:

$$(f + g)_n = f_n + g_n,$$

where we use the fact that $\mathbf{Ab}$ is preadditive. Note that $f + g$ is also a chain map, since it commutes with the boundary operators. The bilinearity of composition follows level-wise from the fact that $\mathbf{Ab}$ is preadditive. $\square$

Of course given two preadditive categories $\mathbf{C}$ and $\mathbf{D}$, not every functor will preserve this extra structure.

Definition 2.10. Let $\mathbf{C}$ and $\mathbf{D}$ be two preadditive categories. A functor $F : \mathbf{C} \to \mathbf{D}$ is said to be *additive* if it preserves addition of maps, i.e.:

$$F(f + g) = F(f) + F(g).$$

In other words the functor $F$ induces a group homomorphism: $F : \mathbf{Hom}_{\mathbf{C}}(A, B) \to \mathbf{Hom}_{\mathbf{D}}(FA, FB)$.

[1] However, this generality might not be so interesting from a categorical standpoint, as there is a fully faithful (exact) functor $F : \mathbf{C} \to \mathbf{Ab}$ for any (small) abelian category $\mathbf{C}$, called the *Mitchell embedding* [**Rot09**]. This gives a way to proof categorical statements in $\mathbf{C}$ by proving the statement in $\mathbf{Ab}$.

**2.2. The singular chain complex.** In order to see why we are interested in the construction of homology groups, we will look at an example from algebraic topology. We will see that homology gives a nice invariant for spaces. We will construct a chain complex for any topological space. In this section we will not be very precise, as it will only act as a motivation. However the intuition might be very useful later on, and so pictures are provided to give meaning to this construction.

Definition 2.11. The *topological $n$-simplex* $\Delta^n$, $n \in \mathbb{N}$, is the set

$$\Delta^n = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0 \text{ and } x_0 + \ldots + x_n = 1\} \subseteq \mathbb{R}^{n+1}$$

endowed with the subspace topology.

In particular $\Delta^0$ is simply a point, $\Delta^1$ a line and $\Delta^2$ a solid triangle. There are nice inclusions $\Delta^n \hookrightarrow \Delta^{n+1}$ which we need. For any $n \in \mathbb{N}$ we define:

Definition 2.12. For $i \in \{0, \ldots, n+1\}$ the $i$-th face map $\delta^i : \Delta^n \hookrightarrow \Delta^{n+1}$ is defined as

$$\delta^i(x_0, \ldots, x_n) = (x_0, \ldots, x_{i-1}, 0, x_i, \ldots, x_n) \text{ for all } x \in \Delta^n.$$

Given a space $X$, we will be interested in continuous maps $\sigma : \Delta^n \to X$, such a map is called a *singular $n$-simplex.* Note that if we have a $(n+1)$-simplex $\sigma : \Delta^{n+1} \to X$ we can precompose with a face map to get a $n$-simplex $\sigma \circ \delta^i : \Delta^n \to X$. This is illustrated in Figure 1 for $n = 1$.

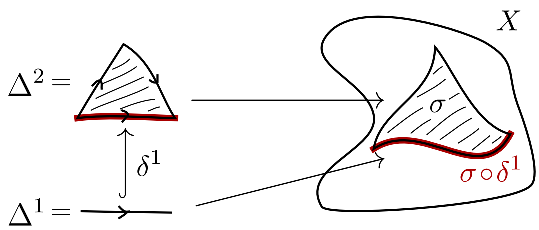


Figure 1. The 2-simplex $\sigma$ gives rise to a 1-simplex $\sigma \circ \delta^1$

From the picture it is clear that the assignment $\sigma \mapsto \sigma \circ \delta^i$ gives one of the faces of the boundary of $\sigma$. We would like to be able to formally add these different $\sigma \circ \delta^i$ in order to assign to $\sigma$ "its complete boundary". This is achieved by passing to free abelian groups as defined in the previous section. However we should note that the topological $n$-simplex is in some way oriented or ordered, which is preserved by the face maps.

Definition 2.13. For a topological space $X$ we define the *$n$-th singular chain group* $C_n(X)$ by

$$C_n(X) = \mathbb{Z}[\mathbf{Hom}_{\mathbf{Top}}(\Delta^n, X)].$$

The *singular boundary operator* $\partial : C_{n+1}(X) \to C_n(X)$ is defined on generators as

$$\partial(\sigma) = \sigma \circ \delta^0 - \sigma \circ \delta^1 + \ldots + (-1)^{n+1} \sigma \circ \delta^{n+1}.$$

The elements in $C_n(X)$ are called *singular $n$-chains* and are formal sums of *singular $n$-simplices*. Since these groups are free, we can define any group homomorphism by defining it on the generators, the $n$-simplices.

Some geometric intuition for the boundary operator is provided by Figure 2. In this picture we see that the boundary of a 1-simplex is simply its end-point minus the starting-point. We see that the boundary of a 2-simplex is an alternating sum of three 1-simplices. The alternating sum ensures that the end-points and starting-points of the resulting 1-chain will cancel out when applying $\partial$ again. So in the degrees 1 and 2 we see that $\partial$ is nicely behaved. We will now claim that this construction indeed gives a chain complex, without proof.

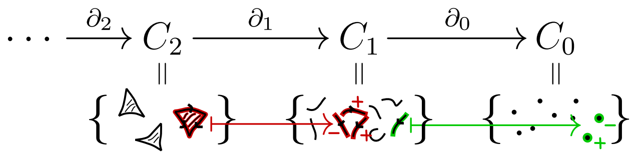


FIGURE 2. The boundary of a 2-simplex, and the boundary of a 1-simplex

The above construction defines a functor $C : \mathbf{Top} \to \mathbf{Ch}(\mathbf{Ab})$ (we will not prove this) which sends a space $X$ to its *singular chain complex* $C(X)$. So the terminology of Definition 2.4 applies to these chain complexes. Composing this with the functor $H_n : \mathbf{Ch}(\mathbf{Ab}) \to \mathbf{Ab}$ gives rise to the following definition.

DEFINITION 2.14. The *$n$-th singular homology group* of a space $X$ is defined as

$$H_n^{\mathrm{sing}}(X) = H_n(C(X)).$$

With Figure 3 we indicate what $H_1^{\mathrm{sing}}$ measures. In the first space $X$ we see a 1-cycle $\sigma_1 - \sigma_2 + \sigma_3$ which is also a boundary, because we can define a map $\tau : \Delta^2 \to X$ such that $\partial(\tau) = \sigma_1 - \sigma_2 + \sigma_3$, hence we conclude that $0 = [\sigma_1 - \sigma_2 + \sigma_3] \in H_1^{\mathrm{sing}}(X)$. So this 1-cycle is not interesting in homology. In the space $X'$ however there is a hole, which prevents a 2-simplex like $\tau$ te exist, hence $0 \neq [\sigma_1 - \sigma_2 + \sigma_3] \in H_1^{\mathrm{sing}}(X')$. This example shows that in some sense this functor is capable of detecting holes in a space.


(A) The 1-cycle is in fact a boundary.

(B) The hole in $X'$ prevents the 1-cycle to be a boundary.

FIGURE 3. Two different spaces in which we consider a 1-chain $\sigma_1 - \sigma_2 + \sigma_3$, this 1-chain is in fact a 1-cycle, because the end-points and starting-points cancel out.

A direct consequence of being a functor is that homeomorphic spaces have isomorphic singular homology groups. There is even a stronger statement which tells us that homotopy equivalent spaces have isomorphic homology groups. So if one is interested in homotopy of a space, then homology already gives some information.

In the remainder of this section we will give the homology groups of some basic spaces. For most spaces it is hard to calculate the homology groups from the definitions above. One generally proves these results by using theorems from algebraic topology or homological algebra, which are beyond the scope of this thesis. The first example can be calculated from the definitions above, however the proof is not included as the example is only included as a motivation.

Example 2.15. The homology of the one-point space $*$ is given by

$$H_n^{\text{sing}}(*) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}.$$

Let $S^k = \{x \in \mathbb{R}^{n+1} \mid ||x|| = 1\}$ be the $k$-sphere. For example, $S^0$ consists only of two points and $S^1$ is the usual circle.

Example 2.16. The homology of $S^k$ for $k > 0$ is given by

$$H_n^{\text{sing}}(S^k) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0 \text{ or } n = k \\ 0 & \text{otherwise} \end{cases}.$$

For $S^0$ the homology group $H_0(S^0)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, and all other homology groups are trivial.

We can use the latter example to prove a fact about $\mathbb{R}^n$ quite easily ($n > 0$). Note that $\mathbb{R}^n - \{0\}$ is homotopy equivalent to $S^{n-1}$, so their homology groups are the same. As a consequence $\mathbb{R}^n - \{0\}$ has the same homology groups as $\mathbb{R}^m - \{0\}$, only if $n = m$. Now if $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^m$, then also $\mathbb{R}^n - \{0\} \cong \mathbb{R}^m - \{0\}$, so this only happens if $n = m$. This result is known as the *invariance of dimension*.

## 3. Simplicial Abelian Groups

Before defining *simplicial abelian groups*, we will first discuss the more general notion of *simplicial sets.* There are generally two definitions of simplicial sets, an abstract one and a very explicit one. We will start with the abstract one, luckily it can still be visualised in pictures, then we will derive the explicit definition. The reader who is interested in how these notions are developed, should consider reading the introduction by Friedman [**Fri12**], which also gives nice illustrations.

### 3.1. Abstract definition.

Definition 3.1. We define a category $\mathbf{\Delta}$, where the objects are the *finite ordinals* $[n] = \{0 < \cdots < n\}$ for $n \in \mathbb{N}$ and maps are *monotone functions*: $\mathbf{Hom}_{\mathbf{\Delta}}([n],[m]) = \{f : [n] \to [m] \mid f(i) \le f(j) \text{ for all } i < j\}$.

The category $\mathbf{\Delta}$ is sometimes referred to as the *category of finite ordinals* or the *cosimplicial index category.* There are two special kinds of maps in $\mathbf{\Delta}$, the so called *face* maps and *degeneracy* maps. The *i-th face maps* $\delta_i : [n-1] \to [n]$ is the unique injective monotone function which *omits* $i$. More precisely, it is defined for all $n \in \mathbb{N}^{>0}$ as (note that we do not explicitly denote $n$ in this notation)

$$\delta_i : [n-1] \to [n], k \mapsto \begin{cases} k & \text{if } k < i, \\ k+1 & \text{if } k \ge i, \end{cases} \qquad 0 \le i \le n.$$

The *i-th degeneracy map* $\sigma_i : [n+1] \to [n]$ is the unique surjective monotone function which *hits* $i$ *twice.* More precisely it is defined for all $n \in \mathbb{N}$ as

$$\sigma_i : [n+1] \to [n], k \mapsto \begin{cases} k & \text{if } k \le i, \\ k-1 & \text{if } k > i, \end{cases} \qquad 0 \le i \le n.$$

The nice things about these maps is that every map in $\mathbf{\Delta}$ can be decomposed to a composition of such maps. So in a sense, these are all the maps we need to consider.

Lemma 3.2. (Epi-mono factorization) *Let* $\eta : [m] \to [n]$ *be a map in* $\mathbf{\Delta}$*. Then* $\eta$ *can be uniquely decomposed as*

$$\eta = \delta_{i_a} \cdots \delta_{i_1} \sigma_{j_b} \cdots \sigma_{j_1},$$

*such that* $0 \le j_b < \cdots < j_1 < m$ *and* $0 \le i_1 < \cdots < i_a \le n$*.*

This is called the *epi-mono factorization*, because it factors any map $\eta$ into a surjective part $(\sigma_{j_b} \cdots \sigma_{j_1})$ and an injective part $(\delta_{i_a} \cdots \delta_{i_1})$. In a diagram:

$$\begin{tikzcd}
{[m]} \arrow[rr, "\eta"] \arrow[dr, two heads, "\sigma_{j_b} \cdots \sigma_{j_1}"'] & & {[n]} \\
& {[k]} \arrow[ur, hook, "\delta_{i_a} \cdots \delta_{i_1}"'] &
\end{tikzcd}$$

Proof. We start with the existence. Consider the set $S = \{k \in [m-1] \mid \eta(k) = \eta(k+1)\}$. These are precisely the elements which are hit twice, now let $S = \{j_1, \ldots, j_{|S|}\}$ with $0 \leq j_{|S|} < \cdots < j_1 < m$. This gives rise to a surjection $\sigma = \sigma_{j_b} \cdots \sigma_{j_1} : [m] \twoheadrightarrow [m - |S|]$.

Similarly consider $T = \{k \in [m - |S|] \mid k \notin \eta[m]\}$. These are precisely the elements which are omitted, now let $T = \{i_1, \ldots, i_{|T|}\}$ with $0 \leq i_1 < \cdots < i_{|T|} \leq n$. This gives an injection $\delta = \delta_{i_a} \cdots \delta_{i_1} : [m - |S|] \hookrightarrow [n]$. Now we see that $\eta = \delta\sigma$.

Now for uniqueness, suppose also $\eta = \delta_{i'_{a'}} \cdots \delta_{i'_1} \sigma_{j'_{b'}} \cdots \sigma_{j'_1}$ such that $0 \leq j'_{b'} < \cdots < j'_1 < m$ and $0 \leq i'_1 < \cdots < i'_{a'} \leq n$. It is immediately clear that $b = b'$ must hold by counting the elements which are hit twice, and therefore also $a = a'$. Note that $\eta(j'_k) = \eta(j'_{k+1})$, because the sequences are ordered in the same way, this means $j_k = j'_k$ for all $k$. Similarly $i_k = i'_k$ for all $k$. $\square$

We can now depict the category $\mathbf{\Delta}$ as in Figure 4. Note that the face and degeneracy maps are not unrelated. We will make the exact relations precise later.

$$\begin{tikzcd}
{[0]} \arrow[r, shift left=2] \arrow[r, shift right=2] & {[1]} \arrow[l] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & {[2]} \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[r, shift left=6] \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[r, shift right=6] & {[3]} \arrow[l, shift left=4] \arrow[l] \arrow[l, shift right=4] & \ldots
\end{tikzcd}$$

Figure 4. The category $\mathbf{\Delta}$ with face and degeneracy maps.

Although this is a very abstract definition, a more geometric intuition can be given. In $\mathbf{\Delta}$ we can regard $[n]$ as an abstract version of the $n$-simplex $\Delta^n$. The face maps $\delta_i$ are then exactly maps which point out how we can embed $[n-1]$ in $[n]$. This is visualized in Figure 5. This picture shows the images of the face maps, for example the image of $\delta_3$ from $[2]$ to $[3]$ is the set $\{0, 1, 2\}$, which corresponds to the bottom face of the tetrahedron. The degeneracy maps are harder to visualize, one can think of them as "collapsing" maps. For example, this collapses a triangle into a line.

This category $\mathbf{\Delta}$ will act as a prototype for these kind of geometric structures in other categories. This leads to the following definition.

Definition 3.3. A *simplicial set* $X$ is a functor

$$X : \mathbf{\Delta}^{op} \to \mathbf{Set}.$$

(Or equivalently a contravariant functor $X : \mathbf{\Delta} \to \mathbf{Set}$.)

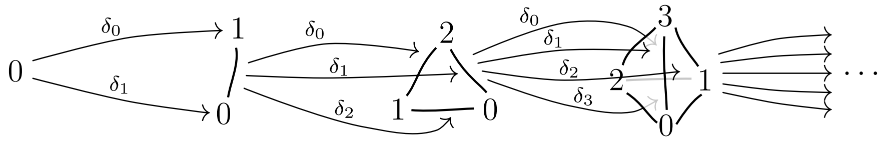


Figure 5. The category $\mathbf{\Delta}$ with the face maps shown in a geometric way.

$$X_0 \;\substack{\longleftarrow\\\longrightarrow\\\longleftarrow}\; X_1 \;\substack{\longleftarrow\\\longrightarrow\\\longleftarrow\\\longrightarrow\\\longleftarrow}\; X_2 \;\substack{\longleftarrow\\\longrightarrow\\\longleftarrow\\\longrightarrow\\\longleftarrow\\\longrightarrow\\\longleftarrow}\; X_3 \dots$$

FIGURE 6. A simplicial set.

The category **sSet** of all simplicial sets is the functor category $\mathbf{Set}^{\mathbf{\Delta}^{op}}$, where morphisms are natural transformations. Because the face and degeneracy maps give all the maps in $\mathbf{\Delta}$ it is sufficient to define images of $\delta_i$ and $\sigma_i$ in order to define a functor $X : \mathbf{\Delta}^{op} \to \mathbf{Set}$, keeping in mind that these should satisfy some relations which we will discuss next. Hence we can depict a simplicial set as done in Figure 6. Comparing this to Figure 4 we see that the arrows are reversed, because $X$ is a contravariant functor.

**3.2. Explicit definition.** Of course the maps $\delta_i$ and $\sigma_i$ in $\mathbf{\Delta}$ satisfy certain relations, these are the so called *cosimplicial identities*.

LEMMA 3.4. *The face and degeneracy maps in* $\mathbf{\Delta}$ *satisfy the* cosimplicial identities*:*

$$\delta_j\delta_i = \delta_i\delta_{j-1}, \qquad \text{if } i < j, \tag{1}$$

$$\sigma_j\delta_i = \delta_i\sigma_{j-1}, \qquad \text{if } i < j, \tag{2}$$

$$\sigma_j\delta_j = \sigma_j\delta_{j+1} = \mathbf{id}, \tag{3}$$

$$\sigma_j\delta_i = \delta_{i-1}\sigma_j, \qquad \text{if } i > j+1, \tag{4}$$

$$\sigma_j\sigma_i = \sigma_i\sigma_{j+1}, \qquad \text{if } i \leq j. \tag{5}$$

PROOF. This follows immediately from the definitions. □

Note that these cosimplicial identities are "purely categorical", i.e. they only use compositions and identity maps. Because a simplicial set $X$ is a contravariant functor, dual versions of these equations hold in its image. For example, the first equation corresponds to $X(\delta_i)X(\delta_j) = X(\delta_{j-1})X(\delta_i)$ for $i < j$. This can be used for an explicit definition of simplicial sets. In this definition a simplicial set $X$ consists of a collection of sets $X_n$ together with face and degeneracy maps. More precisely:

LEMMA 3.5. *A simplicial set $X$ is equivalently specified by a collection sets $X_n$, $n \in \mathbb{N}$, together with functions $d_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$ for $0 \leq i \leq n$ and $n \in \mathbb{N}$, such that the* simplicial identities *hold:*

$$d_id_j = d_{j-1}d_i, \qquad \textit{if } i < j, \tag{6}$$

$$d_is_j = s_{j-1}d_i, \qquad \textit{if } i < j, \tag{7}$$

$$d_js_j = d_{j+1}s_j = \mathbf{id}, \tag{8}$$

$$d_is_j = s_jd_{i-1}, \qquad \textit{if } i > j+1, \tag{9}$$

$$s_is_j = s_{j+1}s_i, \qquad \textit{if } i \leq j. \tag{10}$$

It is already indicated that a functor from $\mathbf{\Delta}^{op}$ to **Set** is determined when the images for the face and degeneracy maps in $\mathbf{\Delta}$ are provided. So this gives a way of restoring the definition from this specification. Conversely, we can apply functoriality to obtain this specification from the definition. We will not give the proof in more detail. From now on we will use the following notation for a simplicial set $X$:

$$X_n = X([n]), \quad s_i = X(\sigma_i) \quad \text{and} \quad d_i = X(\delta_i).$$

For any other map $\beta : [n] \to [p]$ we will denote the induced map by $\beta^* : X_p \to X_n$.

When using a simplicial set to construct another object, it is often handy to use this second definition, as it gives you a very concrete objects to work with. On the other hand, constructing this might be hard (as you would need to provide a lot of details), in this case we will often use the more abstract definition.

Note that because of the third equation, the degeneracy maps $s_i$ are injective. This means that in the set $X_{n+1}$ there are always "copies" of elements of $X_n$. In a way these elements are not interesting, hence we call them degenerate.

Definition 3.6. An element $x \in X_{n+1}$ is *degenerate* if it lies in the image of $s_i : X_n \to X_{n+1}$ for some $i$, otherwise it is called *non-degenerate.*

Lemma 3.7. *We can write any $x \in X_n$ uniquely as $x = \beta^* y$ with $\beta : [n] \twoheadrightarrow [m]$ a surjective map and $y \in X_m$ non-degenerate.*

Proof. We will proof the existence by induction over $n$. For $n = 0$ the statement is trivial, since all elements in $X_0$ are non-degenerate. Assume the statement is proven for $n$. Let $x \in X_{n+1}$. Clearly if $x$ itself is non-degenerate, we can write $x = \mathbf{id}^* x$. Otherwise it is of the form $x = s_i x'$ for some $x' \in X_n$ and $i$. The induction hypothesis tells us that we can write $x' = \beta^* y$ for some surjection $\beta : [n] \twoheadrightarrow [m]$ and $y \in X_m$ non-degenerate. So $x = s_i \beta^* y = (\beta\sigma_i)^* y$.

For uniqueness, assume $x = \beta^* y = \gamma^* z$ with $\beta : [n] \twoheadrightarrow [m]$, $\gamma : [n] \twoheadrightarrow [m']$ and $y \in X_m, z \in X_{m'}$ non-degenerate. Because $\beta$ is surjective there is an $\alpha : [m] \to [n]$ such that $\beta\alpha = \mathbf{id}$ and hence $y = \alpha^*\beta^* y = \alpha^*\gamma^* z = (\gamma\alpha)^* z$. By the epi-mon factorization (Lemma 3.2) we can write $\gamma\alpha = \delta_{i_a} \cdots \delta_{i_1} \sigma_{j_b} \cdots \sigma_{j_1}$, using that $y$ is non-degenerate we know that $\gamma\alpha$ is injective. So we have $\gamma\alpha : [m] \hookrightarrow [m']$. Because of symmetry (of $y$ and $z$) we also have some map $[m'] \hookrightarrow [m]$, so $m = m'$. So $\gamma\alpha$ is also surjective, hence the identity function, thus $y = z$, meaning that the non-degenerate $m$-simplex $y$ is unique.

Now assume $x = \beta^* y = \gamma^* y$ with $\gamma, \beta : [n] \twoheadrightarrow [m]$ such that $\beta \neq \gamma$, and $y \in X_m$ non-degenerate. Then we can find an $\alpha : [m] \to [n]$ such that $\beta\alpha = \mathbf{id}$ and $\gamma\alpha \neq \mathbf{id}$. With the epi-mono factorization write $\gamma\alpha = \delta_{i_a} \cdots \delta_{i_1} \sigma_{j_b} \cdots \sigma_{j_1}$, then by functoriality of $X$

$$y = \alpha^*\beta^* y = \alpha^*\gamma^* y = s_{j_1} \cdots s_{j_b} d_{i_1} \cdots d_{i_a} y.$$

Note that $y$ was non-degenerate, so $s_{j_1} \cdots s_{j_b} = \mathbf{id}$, hence $d_{i_1} \cdots d_{i_a} = \mathbf{id}$. So $\gamma\alpha = \mathbf{id}$, which gives a contradiction. So $\beta = \gamma$, meaning that the surjection $\beta$ is also unique. □

**3.3. The standard $n$-simplex.** Recall that for any category $\mathbf{C}$ we have the $\mathbf{Hom}$-functor $\mathbf{Hom}_{\mathbf{C}}(-,-) : \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{Set}$. We can fix an object $C \in \mathbf{C}$ and get a functor $\mathbf{Hom}_{\mathbf{C}}(-, C) : \mathbf{C}^{op} \to \mathbf{Set}$. In our case we can get the following simplicial sets in this way:

Definition 3.8. The *standard n-simplex* $\Delta[n] \in \mathbf{sSet}$ is given by

$$\Delta[n] = \mathbf{Hom}_{\mathbf{\Delta}}(-, [n]) : \mathbf{\Delta}^{op} \to \mathbf{Set}.$$

Note that $\Delta[-] : \mathbf{\Delta} \to \mathbf{sSet}$ is exactly the Yoneda embedding. So a $m$-simplex in $\Delta[n]$ is nothing more than a monotone function $[m] \to [n]$. In a moment we will see why the Yoneda lemma is useful to us, but let us first describe which functions are non-degenerate. Recall that a simplex is degenerate if it lies the image of $s_i$ for some $i$. In the simplicial set $\Delta[n]$ the degeneracy maps $s_i$ are given by precomposing with $\sigma_i$ (by definition of the **Hom**-functor).

Lemma 3.9. *The non-degenerate $m$-simplices in $\Delta[n]$ are precisely injective monotone functions $[m] \hookrightarrow [n]$.*

Proof. Given a $m$-simplex $x \in \Delta[n]_m$, using the epi-mono factorization we can write it as $x = \delta\sigma : [m] \to [n]$, where $\delta$ is injective and $\sigma$ surjective. It is now easily seen that $x$ is degenerate if and only if $\sigma \neq \mathbf{id}$. In other words a $m$-simplex $x \in \Delta[n]_m$ is non-degenerate if and only if $x : [m] \to [n]$ is injective. Note that for $m > n$ no such injective monotone functions exist and for $m = n$ there is a unique one, namely $\mathbf{id}_{[n]}$. $\square$

Example 3.10. We will compute how $\Delta[0]$ looks like. Note that $[0]$ is an one-element set, so for any set $S$, there is only one function $* : S \to [0]$. Hence $\Delta[0]_n = \{*\}$ for all $n$ and the face and degeneracy maps are necessarily the identity maps $\mathbf{id} : \{*\} \to \{*\}$. Thus, $\Delta[0]$ looks like

$$\Delta[0] := \quad \{*\} \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow} \{*\} \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow} \{*\} \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow} \cdots \ .$$

Note that the only non-degenerate simplex is the unique 0-simplex.

Example 3.11. $\Delta[1]$ is a bit more interesting, but still not too complicated. We will describe the first three sets $\Delta[1]_0$, $\Delta[1]_1$ and $\Delta[1]_2$. We can use the fact that any monotone function $f : [n] \to [m]$ is a composition of first applying degeneracy maps, and then face maps, i.e.: $f : [n] \xrightarrow{\sigma_{i_0} \cdots \sigma_{i_M}} [k] \xrightarrow{\delta_{j_0} \cdots \delta_{j_N}} [m]$, where $k \leq m, n$.

For $\Delta[1]_0$ we have to consider maps from $[0]$ to $[1]$, we cannot first apply degeneracy maps (there is no object $[-1]$). So this leaves us with the face maps: $\Delta[1]_0 = \{\delta_0, \delta_1\}$. For $\Delta[1]_1$ we of course have the identity function and two functions $\delta_0\sigma_0, \delta_1\sigma_0$. Now $\Delta[1]_2$ are the maps from $[2]$ to $[1]$.

We will compute the two face maps $d_0$ and $d_1$ from $\Delta[1]_1$ to $\Delta[1]_0$. Recall that the **Hom**-functor in the first argument (the contravariant argument) works with precomposition. So this gives

$$\begin{aligned} d_0(\mathbf{id}) &= \mathbf{id}\delta_0 = \delta_0 \\ d_0(\delta_0\sigma_0) &= \delta_0\sigma_0\delta_0 = \delta_0 \\ d_0(\delta_1\sigma_0) &= \delta_0\sigma_0\delta_0 = \delta_1. \end{aligned}$$

Where we in the first calculation used the identity law. In the second and third line we used the third simplicial equation, asserting that $\sigma_0\delta_0 = \mathbf{id}$. Similarly we can calculate the face map $d_1$:

$$\begin{aligned} d_1(\mathbf{id}) &= \mathbf{id}\delta_1 = \delta_1 \\ d_1(\delta_0\sigma_0) &= \delta_0\sigma_0\delta_1 = \delta_0 \\ d_1(\delta_1\sigma_0) &= \delta_0\sigma_0\delta_1 = \delta_1. \end{aligned}$$

$$\begin{tikzcd} \Delta[1] := \quad \{\delta_0, \delta_1\} \arrow[r] & \{\delta_0\sigma_0, \mathbf{id}, \delta_1\sigma_0\} \arrow[l, shift right=2] \arrow[l, shift left=2] \arrow[r, shift left=1] \arrow[r, shift right=1] & \cdots \arrow[l, shift right=3] \arrow[l] \arrow[l, shift left=3] \end{tikzcd} \; .$$

In this simplicial set there are three non-degenerate simplices. There is $\mathbf{id} \in \Delta[1]_1$, which clearly is non-degenerate, and the two 0-simplices $\delta_0$ and $\delta_1$. One can think of this simplicial set as a line (the non-degenerate 1-simplex) with its endpoints (the two 0-simplices).

**3.4. Simplicial objects in arbitrary categories.** Of course the definition of simplicial set can easily be generalized to other categories. For any category $\mathbf{C}$ we can consider the functor category $\mathbf{sC} = \mathbf{C}^{\mathbf{\Delta}^{op}}$. In this thesis we are interested in the category of *simplicial abelian groups*:

$$\mathbf{sAb} = \mathbf{Ab}^{\mathbf{\Delta}^{op}}.$$

So a simplicial abelian group $A$ is a collection of abelian groups $A_n$, together with face and degeneracy maps, which in this case means group homomorphisms $d_i$ and $s_i$ such that the simplicial equations hold.

Note that the set of natural transformations between two simplicial abelian groups $A$ and $B$ is also an abelian group. The proof that $\mathbf{sAb}$ is a preadditive category is very similar to the proof we saw in Section 2. For two natural transformations $f, g : A \to B$ we simply define $f + g$ pointwise by $(f + g)_n = f_n + g_n$ and it is easily checked that this is a natural transformation.

As we are interested in simplicial abelian groups, it would be nice to obtain simplicial abelian groups associated to the standard $n$-simplices. We have seen how to make an abelian group out of any set using the free abelian group functor. We can use this functor $\mathbb{Z}[-] : \mathbf{Set} \to \mathbf{Ab}$ to induce a functor $\mathbb{Z}^*[-] : \mathbf{sSet} \to \mathbf{sAb}$ as shown in the following diagram. This construction obviously defines a functor $\mathbb{Z}^*[-] : \mathbf{sSet} \to \mathbf{sAb}$. Similarly,

$$\begin{tikzcd} \mathbf{\Delta}^{op} \arrow[r, "X"] \arrow[dr, "\mathbb{Z}^*[X]"'] & \mathbf{Set} \arrow[d, "\mathbb{Z}[-]"] \\ & \mathbf{Ab} \end{tikzcd}$$

FIGURE 7. The simplicial set $X$ can be made into a simplicial abelian group $\mathbb{Z}^*[X]$ by postcomposing with $\mathbb{Z}[-]$.

postcomposition with the forgetful functor $U : \mathbf{Ab} \to \mathbf{Set}$ gives rise to a forgetful functor $U^* : \mathbf{sAb} \to \mathbf{sSet}$. Thus in formulas we have

$$\mathbb{Z}^*[X]_n = \mathbb{Z}[X_n] \quad \text{and} \quad U^*(A)_n = U(A_n).$$

This justifies that we may drop this extra decoration $(^*)$ and write $\mathbb{Z}[-]$ (resp. $U$) instead of $\mathbb{Z}^*[-]$ (resp. $U^*$).

LEMMA 3.12. *The functor* $\mathbb{Z}[-] : \mathbf{sSet} \to \mathbf{sAb}$ *is a left adjoint, with* $U : \mathbf{sAb} \to \mathbf{sSet}$ *as right adjoint.*

As this is a purely categorical question (it even works for arbitrary functor categories), only a sketch of the proof is given. First note that by the fact that $\mathbb{Z}$ and $U$ already form an adjunction, and if we are given a natural transformation $f : X \to UA$ of simplicial sets we get the following diagram for each $n \in \mathbb{N}$:

$$\begin{tikzcd} X_n \arrow[r, "\eta_{X_n}"] \arrow[dr, "f_n"'] & U\mathbb{Z}[X]_n \arrow[d, "U(\overline{f}_n)"] & \mathbb{Z}[X_n] \arrow[d, "\overline{f}_n"] \\ & U(A_n) & A_n \end{tikzcd}$$

Then use naturality of $\eta$ (in $X_n$, thus in particular in $n$) to extend this to $\eta : X \to U\mathbb{Z}[X]$. The uniqueness of the maps $\overline{f}_n$ will assure that we get a natural transformation $\overline{f} : \mathbb{Z}[X] \to A$. The reader is invited to check the details.

EXAMPLE 3.13. We can apply this to the standard $n$-simplex $\Delta[1]$. This gives $\Delta[1]_0 \cong \mathbb{Z}^2$, since $\Delta[1]_0$ has two elements, and $\mathbb{Z}^*[\Delta[1]]_1 \cong \mathbb{Z}^3$, where the isomorphisms are taken such that

$$\begin{aligned} \delta_0 &\overset{\cong}{\longmapsto} (1,0), \\ \delta_1 &\overset{\cong}{\longmapsto} (0,1), \\ \delta_0\sigma_0 &\overset{\cong}{\longmapsto} (1,0,0), \\ \mathbf{id} &\overset{\cong}{\longmapsto} (0,1,0), \\ \delta_1\sigma_0 &\overset{\cong}{\longmapsto} (0,0,1). \end{aligned}$$

The face maps from $\mathbb{Z}[\Delta[1]]_1$ to $\mathbb{Z}[\Delta[1]]_0$ under these isomorphisms are then given by

$$\begin{aligned} d_0(x,y,z) &= (x+y,z), \\ d_1(x,y,z) &= (x,y+z). \end{aligned}$$

**3.5. The Yoneda lemma.** Recall the statement of the Yoneda lemma from Section 1. In our case we consider functors $X : \mathbf{\Delta}^{op} \to \mathbf{Set}$ and objects $[n]$. So this gives us a natural bijection

$$\mathbf{Hom}_{\mathbf{sSet}}(\Delta[n], X) \cong X_n$$

telling us that we can regard $n$-simplices in $X$ as maps from $\Delta[n]$ to $X$. This also extends to the case of simplicial abelian groups.

LEMMA 3.14. (The additive Yoneda lemma) *Let $A$ be a simplicial abelian group. Then there is a group isomorphism*

$$\mathbf{Hom}_{\mathbf{sAb}}(\mathbb{Z}[\Delta[n]], A) \cong A_n,$$

*which is natural in $A$ and $[n]$.*

PROOF. By using the (non-additive) Yoneda lemma and the fact that $\mathbb{Z}$ is a left adjoint, we already have a natural bijection:

$$\mathbf{Hom}_{\mathbf{sAb}}(\mathbb{Z}[\Delta[n]], A) \cong \mathbf{Hom}_{\mathbf{sSet}}(\Delta[n], U(A)) \cong U(A)_n = A_n.$$

The only thing that we need to check is that this bijection preserves the group structure. Recall that this bijection from $\mathbf{Hom}_{\mathbf{sAb}}(\mathbb{Z}[\Delta[n]], A)$ to $A_n$ is given by (where $\mathbf{id} = \mathbf{id}_{[n]}$ is a generator in $\mathbb{Z}[\Delta[n]]$)

$$\phi(f) = f_n(\mathbf{id}) \in X_n \quad \text{ for } f : \Delta[n] \to X.$$

Now let $A$ be a simplicial abelian group and $f, g : \mathbb{Z}\Delta[n] \to A$ maps. Then we compute

$$\phi(f) + \phi(g) = f_n(\mathbf{id}) + g_n(\mathbf{id}) = (f_n + g_n)(\mathbf{id}) = (f + g)_n(\mathbf{id}) = \phi(f + g),$$

where we regard $\mathbf{id} \in \Delta[n]$ as an element $\mathbf{id} \in \mathbb{Z}\Delta[n]$, we can do so by the unit of the adjunction. So this bijection is also a group homomorphism, hence we have an isomorphism $\mathbf{Hom}_{\mathbf{sAb}}(\mathbb{Z}[\Delta[n]], A) \cong A_n$ of abelian groups. $\square$

## 4. The Dold-Kan correspondence

Comparing chain complexes and simplicial abelian groups, one sees a certain similarity. Both concepts are defined as sequences of abelian groups with certain structure maps. At first sight simplicial abelian groups seem to have a richer structure. There are many face maps as opposed to only a single boundary operator. Nevertheless, as we will show in this section, these two concepts give rise to equivalent categories.

**4.1. Unnormalized chain complex.** Given a simplicial abelian group $A$, we have a family of abelian groups $A_n$. For every $n > 0$ we define a group homomorphism

$$\partial_n = d_0 - d_1 + \ldots + (-1)^n d_n : A_n \to A_{n-1}.$$

LEMMA 4.1. *Using $A_n$ as the family of abelian groups and the maps $\partial_n$ as boundary operators gives a chain complex.*

PROOF. We already have a collection of abelian groups together with maps, so the only thing to prove is $\partial_{n-1} \circ \partial_n = 0$. This can be done with a calculation.

$$\begin{aligned}
\partial_{n-1} \circ \partial_n &= \sum_{i=0}^{n-1} \sum_{j=0}^{n} (-1)^{i+j} d_i \circ d_j \\
&\overset{(1)}{=} \sum_{i=0}^{n-1} \sum_{j=0}^{i} (-1)^{i+j} d_i \circ d_j + \sum_{i=0}^{n-1} \sum_{j=i+1}^{n} (-1)^{i+j} d_i \circ d_j \\
&\overset{(2)}{=} \sum_{i=0}^{n-1} \sum_{j=0}^{i} (-1)^{i+j} d_i \circ d_j + \sum_{i=0}^{n-1} \sum_{j=i+1}^{n} (-1)^{i+j} d_{j-1} \circ d_i \\
&\overset{(3)}{=} \sum_{i=0}^{n-1} \sum_{j=0}^{i} (-1)^{i+j} d_i \circ d_j - \sum_{i=0}^{n-1} \sum_{j=i}^{n-1} (-1)^{i+j} d_j \circ d_i \\
&= \sum_{i=0}^{n-1} \sum_{j=0}^{i} (-1)^{i+j} d_i \circ d_j - \sum_{i=0}^{n-1} \sum_{j=0}^{i} (-1)^{i+j} d_i \circ d_j = 0
\end{aligned}$$

In this calculation we did the following. We split the inner sum in two halves (1) and we use the simplicial equations on the second sum (2). Then we do a shift of indices (3). By interchanging the roles of $i$ and $j$ in the second sum, we have two equal sums which cancel out. So indeed this is a chain complex. □

Thus, associated to a simplicial abelian group $A$ we obtain a chain complex $M(A)$ with $M(A)_n = A_n$ and the boundary operators as above. Following the book [**GJ99**] we will call the chain complex $M(X)$ the *Moore complex* or *unnormalized chain complex* of $X$. This construction defines a functor

$$M : \mathbf{sAb} \to \mathbf{Ch}(\mathbf{Ab})$$

by assigning $M(f)_n = f_n$ for a natural transformation $f : A \to B$. It follows from a nice calculation that $M(f)$ is indeed a chain map:

$$\begin{aligned} f_{n-1} \circ \partial &= f_{n-1} \circ (d_0 - d_1 + \ldots + (-1)^n d_n) \\ &= f_{n-1} \circ d_0 - f_{n-1} \circ d_1 + \ldots + (-1)^n f_{n-1} \circ d_n \\ &\overset{(1)}{=} d_0 \circ f_n - d_1 \circ f_n + \ldots + (-1)^n d_n \circ f_n \\ &= (d_0 - d_1 + \ldots + (-1)^n d_n) \circ f_n = \partial \circ f_n, \end{aligned}$$

where we used naturality of $f$ in step (1). This functor is in fact already used in the construction of the singular chain complex, where we defined the boundary operators (on generators) as $\partial(\sigma) = \sigma \circ d_0 - \sigma \circ d_1 + \ldots + (-1)^{n+1} \sigma \circ d_{n+1}$. We will briefly come back to this in Section 5.

Let us investigate whether this functor $M$ can be part of an equivalence. If $M$ would be part of an equivalence, it would be *essentially surjective*, meaning that for any chain complex $C$ there exists a simplicial abelian group $A$ such that $M(A) \cong C$. For example take the following chain complex

$$C = \ldots \to 0 \to 0 \to \mathbb{Z}.$$

If we want $M$ to be essentially surjective, there should exist a simplicial abelian group $A$ with $A_0 \cong \mathbb{Z}$ and $A_0 \cong 0$. Recall that the degeneracy maps are injective. This contradicts as there is no injective map $\mathbb{Z} \hookrightarrow 0$. So it is easily seen that $M$ cannot be part of an equivalence, although it is a nice functor.

**4.2. Normalized chain complex.** To repair this defect we should be more careful. Given a simplicial abelian group, simply taking the same collection for our chain complex will not work. Instead we are after some “smaller” abelian groups, and in some cases the abelian groups should completely vanish (as in the example above).

Given a simplicial abelian group $A$, we define abelian groups $N(A)_n$ as

$$\begin{aligned} N(A)_n &= \bigcap_{i=1}^{n} \ker(d_i : A_n \to A_{n-1}), \quad n > 0 \\ N(A)_0 &= A_0. \end{aligned}$$

Now define group homomorphisms $\partial : N(A)_n \to N(A)_{n-1}$ as

$$\partial = d_0|_{N(A)_n}.$$

LEMMA 4.2. *The function $\partial$ is well-defined. Furthermore $\partial \circ \partial = 0$.*

PROOF. Let $x \in N(A)_n$, then $d_i \partial(x) = d_i d_0(x) = d_0 d_{i+1}(x) = d_0(0) = 0$ for all $i < n$. So indeed $\partial(x) \in N(A)_{n-1}$, because in particular it holds for $i > 0$. Using this calculation for $i = 0$ shows that $\partial \circ \partial = 0$. This shows that $N(A)$ is a chain complex. $\square$

The chain complex $N(A)$ is called the *normalized chain complex* of $A$.

LEMMA 4.3. *The above construction defines a functor $N : \mathbf{sAb} \to \mathbf{Ch}(\mathbf{Ab})$. Furthermore $N$ is additive.*

PROOF. Given a map $f : A \to B$ of simplicial abelian groups, we consider the restrictions

$$f_n|_{N(A)_n} : N(A)_n \to B_n.$$

Because $f_n$ commutes with the face maps we get

$$d_i(f_n(x)) = f_{n-1}(d_i(x)) = 0,$$

for $i > 0$ and $x \in N(A)_n$. So the restriction also restricts the codomain, in other words $f_n|_{N(A)_n} : N(A)_n \to N(B)_n$ is well-defined. Furthermore it commutes with the boundary operators, since $f$ itself commutes with all face maps. This gives functoriality $N(f) : N(A) \to N(B)$.

Let $f, g : A \to B$ be two maps, then we prove additivity by

$$N(f+g) = (f+g)|_{N(A)} = f|_{N(A)} + g|_{N(A)} = N(f) + N(g).$$

$\square$

EXAMPLE 4.4. We will look at the normalized chain complex of $\mathbb{Z}[\Delta[0]]$. Recall that it looks like

$$\mathbb{Z}[\Delta[0]] := \quad \mathbb{Z} \rightleftarrows \mathbb{Z} \rightleftarrows \mathbb{Z} \rightleftarrows \cdots \ .$$

where all face and degeneracy maps are identity maps. Clearly the kernel of $\mathbf{id}$ is the trivial group. So $N(\mathbb{Z}[\Delta[0]])_i = 0$ for all $i > 0$. In degree zero we are left with $N(\mathbb{Z}[\Delta[0]])_0 = \mathbb{Z}$. So we can depict the normalized chain complex by

$$N(\mathbb{Z}[\Delta[0]]) = \cdots \to 0 \to 0 \to \mathbb{Z}.$$

So in this example we see that the normalized chain complex is really better behaved than the unnormalized chain complex given by $M(\mathbb{Z}[\Delta[0]])$.

To see what $N$ exactly does there are some useful lemmas. These lemmas can also be found in [**Lam68**, Chapter VIII 1-2], but in this thesis more detail is provided. Some corollaries are provided to give some intuition, or so summarize the lemmas, these results can also be found in [**Wei94**, Chapter 8.2-4]. For the following lemmas let $X \in \mathbf{sAb}$ be an arbitrary simplicial abelian group and $n \in \mathbb{N}$. For these lemmas we will need the subgroups $D_n(X) \subseteq X_n$ of degenerate simplices, defined as:

$$D_n(X) = \sum_{i=0}^{n} s_i(X_{n-1}).$$

LEMMA 4.5. *For all $x \in X_n$ we have:*

$$x = b + c,$$

*where $b \in N(X)_n$ and $c \in D_n(X)$.*

PROOF. Define the subgroup $P_n^k = \{x \in X_n \mid d_i x = 0 \text{ for all } i > k\}$. Note that by definition we have

$$N(X)_n = P_n^0 \subseteq P_n^1 \subseteq \ldots \subseteq P_n^{n-1} \subseteq P_n^n = X_n.$$

We will prove with induction that for any $k \leq n$ we can write $x \in X_n$ as $x = b + c$, with $b \in P_n^k$ and $c \in D_n(X)$. For $k = n$ the statement is clear, because we can simply write $x = x$, knowing that $x \in P_n^n = X_n$.

Assume the statement holds for $k > 0$, we will prove it for $k - 1$. So for any $x \in X_n$ we have $x = b + c$, with $b \in P_n^k$ and $c \in D_n(X)$. Now consider $b' = b - s_{k-1}d_k b$. Now clearly for all $i > k$ we have $d_i b' = 0$. For $k$ itself we can calculate

$$d_k(b') = d_k(b - s_{k-1}d_k b) = d_k b - d_k s_{k-1} d_k b = d_k b - d_k b = 0,$$

where we used the equality $d_k s_{k-1} = \mathbf{id}$. So $b' \in P_n^{k-1}$. Furthermore we can define $c' = s_{k-1}d_k b + c$, for which it is clear that $c' \in D_n(X)$. Finally conclude that

$$x = b + c = b - s_{k-1}d_k b + s_{k-1}d_k b + c = b' + c',$$

with $b' \in P_n^{k-1}$ and $c' \in D_n(X)$.

Doing this inductively gives us $x = b + c$, with $b \in P_n^0 = N(X)_n$ and $c \in D_n(X)$. $\square$

LEMMA 4.6. *For all $x \in X_n$, if $s_i x \in N(X)_{n+1}$, then $x = 0$.*

PROOF. Using that $s_i x \in N(X)_{n+1}$ means $0 = d_{k+1}s_i x$ for any $k \geq 0$ and by using using the simplicial identity: $d_{i+1}s_i = \mathbf{id}$, we can conclude $x = d_{i+1}s_i x = 0$. $\square$

The first lemma tells us that every $n$-simplex in $X$ can be decomposed as a sum of something in $N(X)$ and a degenerate $n$-simplex. The latter lemma assures that there are no degenerate $n$-simplices in $N(X)$. So this gives us:

COROLLARY 4.7. $X_n = N(X)_n \oplus D_n(X)$

We can extend the above lemmas to a more general statement.

LEMMA 4.8. *For all $x \in X_n$ we can write $x$ as*

$$x = \sum_\beta \beta^*(x_\beta),$$

*for certain $x_\beta \in N(X)_p$, where $\beta$ ranges over all surjective functions $\beta : [n] \twoheadrightarrow [p]$.*

PROOF. We will proof this using induction on $n$. For $n = 0$ the statement is clear because $N(X)_0 = X_0$.

Assume the statement is proven for $n$. Let $x \in X_{n+1}$, then from Lemma 4.5 we see $x = b + c$. Note that $c \in D_n(X)$, in other words $c = \sum_{i=0}^{n-1} s_i c_i$, with $c_i \in X_n$. So with the induction hypothesis, we can write these as $c_i = \sum_\beta \beta^* c_{i,\beta}$, where the sum quantifies over $\beta : [n] \twoheadrightarrow [p]$. Now $b$ is already in $N(X)_{n+1}$, so we can set $x_{\mathbf{id}} = b$, to obtain the conclusion. $\square$

LEMMA 4.9. *Let $\beta : [n] \twoheadrightarrow [m]$ and $\gamma : [n] \twoheadrightarrow [m']$ be two maps such that $\beta \neq \gamma$. Then we have $\beta^*(N(X))_m \cap \gamma^*(N(X))_{m'} = 0$.*

Proof. Note that $N(X)_i$ only contains non-degenerate $i$-simplices (and 0). For $x \in \beta^*(N(X))_p \cap \gamma^*(N(X))_q$ we have $x = \beta^* y = \gamma^* y'$, where $y$ and $y'$ are non-degenerate. By Lemma 3.7 we know that every $n$-simplex is *uniquely* determined by a non-degenerate simplex and a surjective map. For $x \neq 0$ this gives a contradiction. □

Again the former lemma of these two lemmas proves the existence of a decomposition and the latter shows the uniqueness. So combining these gives:

Corollary 4.10. *For all $x \in X_n$ we can write $x = \sum_\beta \beta^*(x_\beta)$ in a unique way.*

And by considering $X_n$ as a whole we get:

Corollary 4.11. $X_n = \bigoplus_{[n] \twoheadrightarrow [p]} N(X)_p$.

Using Corollary 4.10 we can prove a nice categorical fact about $N$, which we will use later on.

Lemma 4.12. *The functor $N$ is fully faithful, i.e.*

$$N : \mathbf{Hom_{sAb}}(A, B) \cong \mathbf{Hom_{Ch(Ab)}}(N(A), N(B)) \quad A, B \in \mathbf{sAb}.$$

Proof. First we prove that $N$ is injective on maps. Let $f : A \to B$ and assume $N(f) = 0$, for $x \in A_n$ we know $x = \sum_\beta \beta^* x_\beta$, so

$$\begin{aligned} f(x) &= f(\textstyle\sum_\beta \beta^*(x_\beta)) \\ &= \textstyle\sum_\beta f(\beta^*(x_\beta)) \\ &= \textstyle\sum_\beta \beta^*(f(x_\beta)) \\ &= \textstyle\sum_\beta \beta^*(N(f)(x_\beta)) = 0, \end{aligned}$$

where we used naturality of $f$ in the second step, and the fact that $x_\beta \in N(A)$ in the last step. We now see that $f(x) = 0$ for all $x$, hence $f = 0$. So indeed $N$ is injective on maps.

Secondly we have to prove $N$ is surjective on maps. Let $g : N(A) \to N(B)$, define $f : A \to B$ as

$$f(x) = \sum_\beta \beta^* g(x_\beta),$$

again we have written $x$ as $x = \sum_\beta \beta^* x_\beta$. Clearly $N(f) = g$. □

If we reflect a bit on why the functor $M$ was not a candidate for an equivalence, we see that $N$ does a better job. We see that $N$ leaves out all degenerate simplices, so it is more carefully chosen than $M$, which included everything. In fact, Corollary 4.7 exactly tells us $M(X)_n = N(X)_n \oplus D_n(X)$.

**4.3. From $\mathbf{Ch}(\mathbf{Ab})$ to sAb.** In this subsection we will construct a functor from chain complexes to simplicial abelian groups. We will do this in a fairly abstract way. There is, however, also an explicit description of this functor which will be given after proving the main equivalence.

Let $A$ be an additive category and $F : \mathbf{sAb} \to A$ an additive functor. We want to construct a functor $G : A \to \mathbf{sAb}$ which is right adjoint to $F$. For each $a \in A$ we have to specify $G(a) : \mathbf{\Delta}^{op} \to \mathbf{Ab}$. Assume we already specified this, such that $G$ is the right adjoint, then by the additive Yoneda lemma we know

$$\begin{aligned} G(a)_n &\cong \mathbf{Hom}_{\mathbf{sAb}}(\mathbb{Z}[\Delta[n]], G(a)) \\ &\cong \mathbf{Hom}_A(F\mathbb{Z}[\Delta[n]], a). \end{aligned}$$

This in fact can be used as the definition of $G$:

$$G(a)_n = \mathbf{Hom}_A(F\mathbb{Z}[\Delta[n]], a).$$

To check that indeed $G(a) \in \mathbf{sAb}$ we only have to remind ourselves that we only composed two functors, namely

$$\mathbf{\Delta} \xrightarrow{\Delta[-]} \mathbf{sSet} \xrightarrow{\mathbb{Z}} \mathbf{sAb} \xrightarrow{F} A \quad \text{and}$$
$$\mathbf{Hom}_A(-, a) : A^{op} \to \mathbf{Ab}$$

giving us $\mathbf{Hom}_A(F\mathbb{Z}[\Delta[-]], a) : \mathbf{\Delta}^{op} \to \mathbf{Ab}$. Similarly $G$ itself is a functor, because it is defined using the **Hom**-functor.

Many functors to **sAb** can be shown to have this description.[2] In our case we can define a functor $K$ as

$$K : \mathbf{Ch}(\mathbf{Ab}) \to \mathbf{sAb}$$
$$K(C) = \mathbf{Hom}_{\mathbf{Ch}(\mathbf{Ab})}(N\mathbb{Z}[\Delta[-]], C).$$

This is a very abstract definition so we will first discuss what a chain map $N\mathbb{Z}[\Delta[n]] \to C$ looks like. Recall that the non-degenerate $m$-simplices of $\Delta[n]$ are exactly injective maps $\eta : [m] \hookrightarrow [n]$ (Lemma 3.9). So $N\mathbb{Z}[\Delta[n]]$ consists of linear combinations of those non-degenerate simplices, as $N$ precisely gives us the non-degenerate elements. Note that $N\mathbb{Z}[\Delta[n]]_m$ are free groups, since $\mathbb{Z}[\Delta[n]]_m$ are free. In other words, when defining a chain map $N\mathbb{Z}[\Delta[n]] \to C$ it is sufficient to define it on the generators, i.e. on the injections $\eta : [m] \hookrightarrow [n]$. This fact is used throughout the following proofs.

Furthermore the degeneracy maps $s_i : K(C)_{n-1} \to K(C)_n$ are given by precomposition of the induced map $\sigma_{i*} : N\mathbb{Z}[\Delta[n]] \to N\mathbb{Z}[\Delta[n-1]]$ which in their turn are given by postcomposition. More precisely this gives $s_i(f)_m(\eta) = f_m(\sigma_i\eta)$ for any $f \in K(C)_{n-1}$ and $\eta : [m] \hookrightarrow [n]$. We will now have a closer look at the degenerate elements of $K(C)$.

LEMMA 4.13. *Let $f : N\mathbb{Z}[\Delta[n]] \to C$ be a chain map then $f \in D_n(K(C))$ if and only if $f_r = 0$ forall $r \geq n$.*

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[2]And also many functors to **sSet** are of this form if we leave out all additivity requirements.

Proof. If $f \in D_n(K(C))$ we can write $f$ as $f = \sum_{i=0}^{n} s_i(f^{(i)})$ for some maps $f^{(i)} : N\mathbb{Z}[\Delta[n-1]] \to C$. Since $N\mathbb{Z}[\Delta[n-1]]_r = 0$ as there are no injections $[r] \hookrightarrow [n-1]$, we have $f_r^{(i)} = 0$ for all $r > n-1$.

For the other direction let $f : N\mathbb{Z}[\Delta[n]] \to C$ be a chain map and $f_r = 0$ forall $r \geq n$. Define $f_m^{(i)}(\eta) = f_m(\delta_i \eta)$ for $\eta : [m] \hookrightarrow [n]$. This gives a chain map $f^{(i)} : N\mathbb{Z}[\Delta[n-1]] \to C$ by a simple calculation:

$$\partial(f_m^{(i)}(\eta)) = \partial(f_m(\delta_i \eta)) \overset{(1)}{=} f_{m-1}(\partial(\delta_i \eta)) \overset{(2)}{=} f_{m-1}(\delta_i \eta \delta_0) \overset{(2)}{=} f_{m-1}^{(i)}(\partial(\eta)),$$

where we used that $f$ is a chain map at (1) and the definition of the boundary operator of $N(-)$ *and* the definition of face maps in $\Delta[-]$ at (2).

Now let $\eta : [m] \hookrightarrow [n]$ and $\eta \neq \mathbf{id}$ (we already know $f(\mathbf{id}_{[n]}) = 0$ by assumption) then by the epi-mono factorization we have $\eta = \delta_{i_a} \cdots \delta_{i_1}$ with $a > 0$, so

$$f_m(\eta) = f_m(\delta_{i_a} \cdots \delta_{i_1}) \overset{(1)}{=} f_m^{(i_a)}(\delta_{i_{a-1}} \cdots \delta_{i_1}) \overset{(2)}{=} f_m^{(i_a)}(\sigma_{i_a} \delta_{i_a} \delta_{i_a} \cdots \delta_{i_1}) \overset{(3)}{=} s_{i_a}(f^{(i_a)})_m(\eta),$$

where we used the definition of $f_m^{(i_a)}$ at (1), one of the simplicial identities at (2) and the definition of degeneracy maps at (3) as discussed earlier.

By the fact that injections are generators this gives $f_m = \sum_{i=0}^{n} s_i(f^{(i)})_m$ for all $m$, i.e. $f = \sum_{i=0}^{n} s_i(f^{(i)})$. Hence $f \in D_n(K(C))$. □

We now have enough lemmas to prove the main equivalence quite easily. The most important lemma for the isomorphism $X \cong KNX$ will be the lemma stating that $N$ is fully faithful. For the other isomorphism we will use the above lemma to characterize the degenerated simplices in $K(C)$.

Theorem 4.14. *$N$ and $K$ form an equivalence.*

Proof. Let $X$ be a simplicial abelian group. Then we have the following natural isomorphisms of abelian groups:

$$\begin{aligned} X_n &\overset{(1)}{\cong} \mathbf{Hom}(\mathbb{Z}[\Delta[n]], X) \\ &\overset{(2)}{\cong} \mathbf{Hom}(N\mathbb{Z}[\Delta[n]], NX) \\ &\overset{(3)}{=} KN(X)_n \end{aligned}$$

Where we used the additive Yoneda lemma at (1) (Lemma 3.14), then we use the fully faithfulness of $N$ at (2) (Lemma 4.12) and at (3) we simply use the definition of $K$. Using naturality in $n$ we have established $X \cong KNX$ and by naturality in $X$ we have $\mathbf{id} \cong KN$, proving the first part of the equivalence.

For the second part we will explicitly define an isomorphism as

$$\begin{aligned} \phi_n : NK(C)_n &\to C_n \\ f &\mapsto f_n(\mathbf{id}_{[n]}). \end{aligned}$$

Note that this is well defined by the fact that $\mathbf{id}_{[n]}$ is a non-degenerate simplex. This defines a natural chain map, because

$$\phi(\partial(f)) = \partial(f)_{n-1}(\mathbf{id}) \overset{(1)}{=} (f_{n-1} \circ \partial)(\mathbf{id}) \overset{(2)}{=} (\partial \circ f_n)(\mathbf{id}) = \partial(\phi(f)),$$

where we used the definition of $\partial$ at (1) and the fact that $f$ is a chain map at (2). Naturality follows easily by calculating

$$\phi(NK(g)(f)) = \phi(g \circ f) = g_n(f_n(\mathbf{id})) = g_n(\phi(f)).$$

We will first show that $\phi_n$ is surjective. Let $x \in C_n$ define a chain map as

$$\begin{aligned} g_r(y) &= 0 \qquad \text{for } r \neq n, n-1 \\ g_n(\mathbf{id}_{[n]}) &= x \\ g_{n-1}(\delta_i) &= \begin{cases} \partial(x) & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Clearly $\phi_n(g) = x$ by definition and $g$ is a chain map as we defined it to commute with the boundary operators. For proving injectivity consider $g \in \ker(\phi_n)$ then for trivial reasons we have $f_r = 0$ for all $r > n$ and $f_n(\mathbf{id}_{[n]}) = 0$ gives $f_n = 0$. Applying Lemma 4.13 gives us $f \in D_n(K(C))$, but $f \in N(K(C))_n$. So by using Corollary 4.7 we get $f = 0$. Thus $\phi_n$ is an isomorphism, which gives us $NK(C) \cong C$.

We now have established two natural isomorphisms $\mathbf{id}_{\mathbf{sAb}} \cong KN$ and $NK \cong \mathbf{id}_{\mathbf{Ch}}(\mathbf{Ab})$. Hence we have an equivalence $\mathbf{Ch}(\mathbf{Ab}) \simeq \mathbf{sAb}$. □

One might not be content with the abstract description of the functor $K$. In the remainder of this section a more explicit description will be given, and it will be indicated why the two descriptions coincide.

Definition 4.15. For a chain complex $C$ define the abelian groups

$$K'(C)_n = \bigoplus_{\beta} C_p^{\beta},$$

where $\beta$ ranges over all surjections $\beta : [n] \twoheadrightarrow [p]$ and $C_p^{\beta} = C_p$ ($\beta$ only acts as a decoration).

Before we provide the face and degeneracy maps, one should see a nice symmetry with Corollary 4.11. One can also prove the equivalence with this definition. The first isomorphism will be harder to prove, whereas the second isomorphism is easier, as we get the characterization given by Lemma 4.13 almost by definition.

For a chain complex $C$ we will turn the groups $K'(C)_n$ into a simplicial abelian group by defining $K'$ on functions. Let $\alpha : [m] \to [n]$ be a function in $\mathbf{\Delta}$, we will define $K'(\alpha) : K(C)_n \to K(C)_m$ by defining it on each summand $C_p^{\beta}$. Fix a summand $C_p^{\beta}$, by using the epi-mono factorization we know $\beta\alpha = \delta\sigma$ for some injection $\delta$ and some surjection $\sigma$. In the case $\delta = \mathbf{id}$, we make the following identification

$$C_p^{\beta} \xrightarrow{=} C_p^{\sigma} \subset K'(C)_m.$$

In the case $\delta = \delta_0$ we use the boundary operator as follows:

$$C_p^\beta \xrightarrow{\partial} C_{p-1} \xrightarrow{=} C_{p-1}^\sigma \subseteq K'(C)_m.$$

In all the other cases we define the map $C_p^\beta \to K'(C)_m$ to be the zero map. We now have defined a map on each of the summands which gives a map $K'(\alpha) : K'(C)_n \to K'(C)_m$.

We will not show that this functor $K'$ is isomorphic to our functor $K$ defined earlier, however we will indicate that it makes sense by writing out explicit calculations for $K(C)_0$ and $K(C)_1$. First we see that

$$K(C)_0 = \mathbf{Hom}_{\mathbf{Ch(Ab)}}(N\mathbb{Z}^*\Delta[0], C) = \left\{ \begin{array}{r} \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z} \\ \downarrow f_2 \quad \downarrow f_1 \quad \downarrow f_0 \\ \cdots \longrightarrow C_2 \rightarrow C_1 \rightarrow C_0 \end{array} \right\} \cong C_0 = K'(C)_0,$$

because for $f_1, f_2, \ldots$ there is no choice at all, and for $f_0 : \mathbb{Z} \to C_0$ we only have to choose an image for $1 \in \mathbb{Z}$. In the next dimension we see

$$K(C)_1 = \mathbf{Hom}_{\mathbf{Ch(Ab)}}(N\mathbb{Z}^*\Delta[1], C) = \left\{ \begin{array}{r} \cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^2 \\ \downarrow f_2 \quad \downarrow f_1 \quad \downarrow f_0 \\ \cdots \longrightarrow C_2 \rightarrow C_1 \rightarrow C_0 \end{array} \right\} \cong C_1 \oplus C_0 = K'(C)_1,$$

because again we can choose $f_1$ anyway we want, which gives us $C_1$. But then we are forced to choose $f_0(x, x) = \partial(f_1(x))$ for all $x \in \mathbb{Z}$, so we are left with choosing an element $c \in C_0$ for defining $f(1, -1) = c$. Adding this gives $C_1 \oplus C_0$.

## 5. Homotopy

We have already seen homology in chain complexes. We can of course now translate this notion to simplicial abelian groups, by assigning a simplicial abelian group $X$ to $H_n(N(X))$. But there is a more general notion of homotopy for simplicial sets, which is also similar to the notion of homotopy in topology. We will define the notion of homotopy groups for simplicial sets.

When dealing with homotopy groups in a topological space $X$ we always need a base-point $* \in X$. This is also the case for simplicial sets. We will notate the chosen base-point of a simplicial set $X$ with $* \in X_0$. More formally, a *pointed simplicial set* $(X, *)$ is a simplicial set $X$ together with a 0-simplex $* \in X_0$. By the Yoneda lemma this 0-simplex corresponds to a map $\Delta[0] \to X$, and any simplex in the image will be denoted by $*$. Another way of saying this is that we denote the degenerate simplices $s_0(\dots(s_0(*))\dots) \in X_n$ as $*$. Of course in our situation we are concerned with simplicial abelian groups, where there is an obvious choice for the base-point given by the neutral element 0.

### 5.1. Homotopy groups.

Definition 5.1. Given a simplicial set $X$ with base-point $*$, we define $Z_n(X)$ to be the set of $n$-simplices with the base-point as boundary, i.e.

$$Z_n(X) = \{x \in X_n | d_i(x) = * \text{ for all } i \leq n\}.$$

For two $n$-simplices $x, x' \in Z_n(X)$, we define $x \sim x'$ if there exists $y \in X_{n+1}$ such that

$$d_0(y) = x, \tag{11}$$

$$d_1(y) = x', \tag{12}$$

$$d_i(y) = * \text{ for all } i > 1. \tag{13}$$

We will call $y$ the *homotopy* and notate $y : x \sim x'$.

Of course we would like $\sim$ to be an equivalence relation, however this is not true for all simplicial sets. For example there is in general no reason for symmetry, existence of a homotopy from $x$ to $x'$ does not give us a homotopy from $x'$ to $x$. One can give an precise condition on when it is a equivalence relation, the so called *Kan-condition*. In our case of simplicial abelian groups, however, we can prove directly that $\sim$ is an equivalence relation.

In figure 8 it is shown why the definition of homotopy makes sense for $n = 1$. Two homotopic 1-simplices from $Z_n(X)$ are depicted in two ways. The first way only shows the structure we have, indicating what the boundaries are (as described by the face maps). In the second figure we collapsed all occurrences of 0 into a single point. This way of drawing a homotopy should remind the reader of homotopy (between paths) in a topological space.

Lemma 5.2. *For any simplicial abelian group $X$, the relation $\sim$ as defined above is an equivalence relation on $Z_n(X)$. Furthermore it is compatible with addition.*


FIGURE 8. In the figure on the left two homotopic 1 simplices $x, x' \in Z_n(X)$ are shown. The fact that $d_2(y) = *$ is depicted by crossing out the bottom line. The right image shows exactly the same structure if we would draw the 0-simplex 0 only once (and hence also collapse the degenerate 1-simplex $d_2 y$).

Before proving this, one should have a look at figure 9. In this figure we show what we want to proof in degree $n = 0$ (i.e. the simplices of interest are points, and the homotopies are paths).

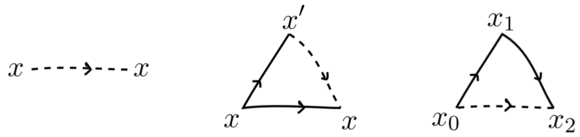


FIGURE 9. The three properties of an equivalence relation: reflexivity, symmetry and transitivity. The dashed lines show which homotopy we should construct.

PROOF. *Reflexivity.* Let $x \in Z_n(X)$, define $y = s_0 x$. By considering the simplicial identities $d_0 s_0 = \mathbf{id}$ and $d_1 s_0 = \mathbf{id}$, it follows that $d_0 y = d_1 y = x$. Furthermore $d_i y = d_i s_0 x = s_0 d_{i-1} x = 0$ for all $i > 1$, because $x \in Z_n(X)$.

*Symmetry.* Let $x, x' \in Z_n(X)$ with $y : x \sim x'$. Define $y' = s_0 x + s_0 x' - y$, then by using linearity: $d_0 y' = x + x' - x = x'$ and $d_1 y' = x + x' - x' = x$. For $i > 1$ we again get $d_i y' = 0$, because $x \in Z_n(X)$.

*Transitivity.* Let $x_0, x_1, x_2 \in Z_n(X)$ with $y : x_0 \sim x_1$ and $z : x_1 \sim x_2$. Define $w = y + z - s_0 x_1$. By linearity we have $d_0 w = x_0 + x_1 - x_1 = x_0$, similarly $d_1 w = x_2$. Again for $i > 1$ we have $d_i w = 0$.

*Addition.* Let $y : x_0 \sim x_1$ and $z : x_2 \sim x_3$. Then by linearity $y + z : x_0 + x_2 \sim x_1 + x_3$ and $-y : -x_0 \sim -x_1$. □

DEFINITION 5.3. Given a simplicial abelian group $X$, we define the $n$-th homotopy group as

$$\pi_n(X) = Z_n(X)/_\sim .$$

Note that this is an abelian group, because $Z_n(X)$ is a subgroup of $X_n$, and $\sim$ also defines a subgroup. It is relatively straight forward to prove that this definition coincides with the $n$-th homology group of the associated normalized chain complex.

Lemma 5.4. *For any simplicial abelian group $X$:*

$$\pi_n(X) = H_n(N(X)).$$

Proof. By writing out the definitions of the $n$-cycles and $n$-boundaries of the normalized chain complex, we see:

$$\begin{aligned}
\ker(\partial) &= \{x \in N(X)_n \mid \partial(x) = 0\} \\
&= \{x \in X_n \mid d_i(x) = 0 \text{ forall } i > 0 \text{ and } d_0(x) = 0\} \\
&= \{x \in X_n \mid d_i(x) = 0 \text{ forall } i \leq n\} \\
&= Z_n(X) \\
\operatorname{im}(\partial) &= \{\partial(y) \mid y \in N(X)_{n+1}\} \\
&= \{d_0 y \mid y \in X_{n+1}, d_i(y) = 0 \text{ for all } i > 0\} \\
&= \{x \in N(X)_n \mid x \sim 0\}
\end{aligned}$$

So we see that $\pi_n(X) = Z_n(X)/_{\sim} = \ker(\partial)/\operatorname{im}(\partial) = H_n(N(X))$. □

Corollary 5.5. *For a chain complex $C$ we have $H_n(C) \cong \pi_n(K(C))$.*

Proof. By the established equivalence we have for any chain complex $C$:

$$\pi_n(K(C)) \cong H_n(N(K(C))) \cong H_n(C).$$

□

**5.2. Topology.** In Section 2, we already defined the topological $n$-simplex $\Delta^n \in$ **Top**. We will now relate these spaces to the standard $n$-simplices $\Delta[n] \in$ **sSet**. We will define a functor $\Delta^- : \mathbf{\Delta} \to \mathbf{Top}$ as follows

$$\begin{aligned}
\Delta^-([n]) = \Delta^n = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0 \text{ and } x_0 + \ldots + x_n = 1\}, \\
\Delta^-(\delta_i)(x_0, \ldots, x_n) = (x_0, \ldots, x_{i-1}, 0, x_i, \ldots, x_n), \\
\Delta^-(\sigma_i)(x_0, \ldots, x_n) = (x_0, \ldots, x_i + x_{i+1}, \ldots, x_n).
\end{aligned}$$

The definition of $\Delta^-(\delta_i)$ was already defined in Section 2 as the face maps $\delta^i : \Delta^n \to \Delta^{n+1}$. So in addition we defined degeneracy maps. The reader is invited to check the cosimplicial identities himself and conclude that we have a functor $\Delta^- : \mathbf{\Delta} \to \mathbf{Top}$. By composing this with the **Hom**-functor we obtain a functor $S : \mathbf{Top} \to \mathbf{sSet}$ given by

$$\operatorname{Sing}(X)_n = \mathbf{Hom}_{\mathbf{Top}}(\Delta^n, X).$$

Recall construction of the singular chain complex in Section 2:

$$C_n(X) = \mathbb{Z}[\mathbf{Hom}_{\mathbf{Top}}(\Delta^n, X)].$$

Where the boundary operator was given as an alternating sum. Looking more closely we see that this construction decomposes as:

$$C : \mathbf{Top} \xrightarrow{\operatorname{Sing}} \mathbf{sSet} \xrightarrow{\mathbb{Z}} \mathbf{sAb} \xrightarrow{M} \mathbf{Ch}(\mathbf{Ab}),$$

where the last functor is the *unnormalized chain complex*. All the categories involved have a notion of homotopy. In topological spaces this is the known notion where $f, g : X \to Y$ are homotopic if there exists a homotopy $H : I \times X \to Y$ with the appropriate properties. In simplicial sets (or simplicial abelian groups) we only saw the notion of homotopy groups, but there exists a more general notion of homotopy, as discussed in the overview of Friedman [**Fri12**]. And finally in chain complexes we saw homology groups, but this category also has a more general notion of chain homotopy, which can be found in any book on homological algebra such as in the book of Rotman [**Rot09**].

It is known that for any simplicial abelian group both the normalized and unnormalized chain complex have the same homology groups. More precisely for any simplicial abelian group $X$ we have:

$$H_n(N(X)) \cong H_n(M(X)) \quad \text{for all } n \in \mathbb{N}.$$

This is for example proven in [**EML53**, Theorem 4.1]. So this assures that the homology groups of the singular chain complex of a space are really the homotopy groups of the simplicial abelian group which is in the background.

## Conclusion

In this thesis we have seen two interesting mathematical structures. On one hand there are simplicial sets and simplicial abelian groups which are defined in a abstract and categorical way. The definition was quite short and elegant, nevertheless the objects have a very rich geometrical structure. On the other hand there are chain complexes which have a very simple definition, which are at first sight completely algebraic.

A proof was given of the equivalence of these structures. In this proof we had to take a close look at degenerated simplices in simplicial abelian groups. Some abstract machinery from category theory, like the Yoneda lemma, allowed us to easily construct the needed isomorphisms.

The category of simplicial sets is the abstract framework for doing homotopy theory. Using free abelian groups allowed us to linearize this, resulting in the category of simplicial abelian groups. The Dold-Kan correspondence assures us that there is no loss of information when passing to chain complexes. This makes the category of chain complexes and homological algebra very suitable for doing homotopy theory in a linearized fashion.

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